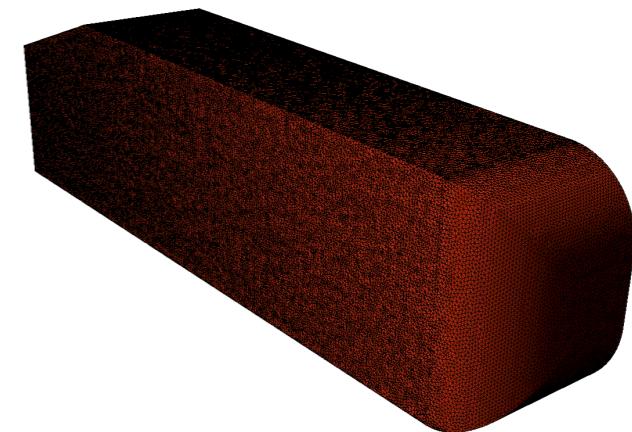
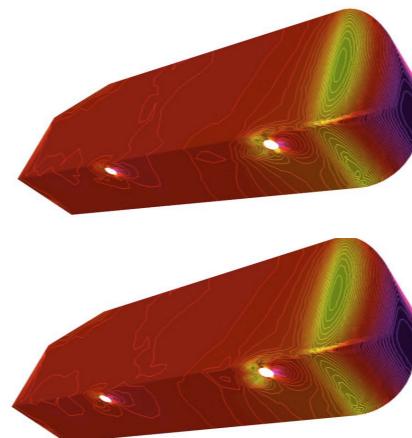
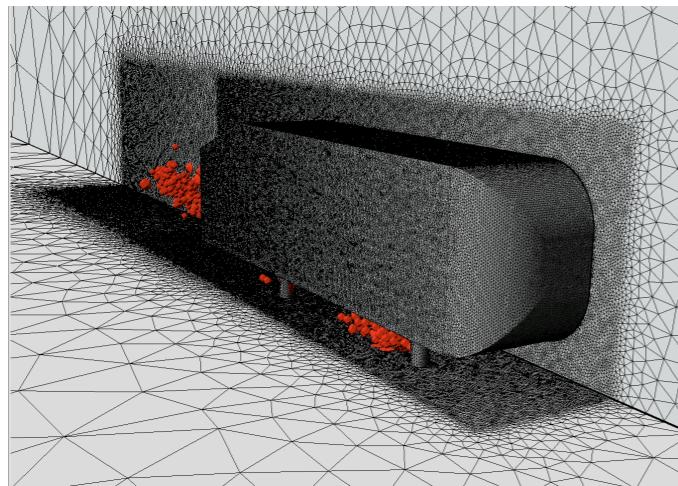


The GNAT method for nonlinear model reduction: recent developments and application to large-scale models



Kevin Carlberg
Sandia National Laboratories

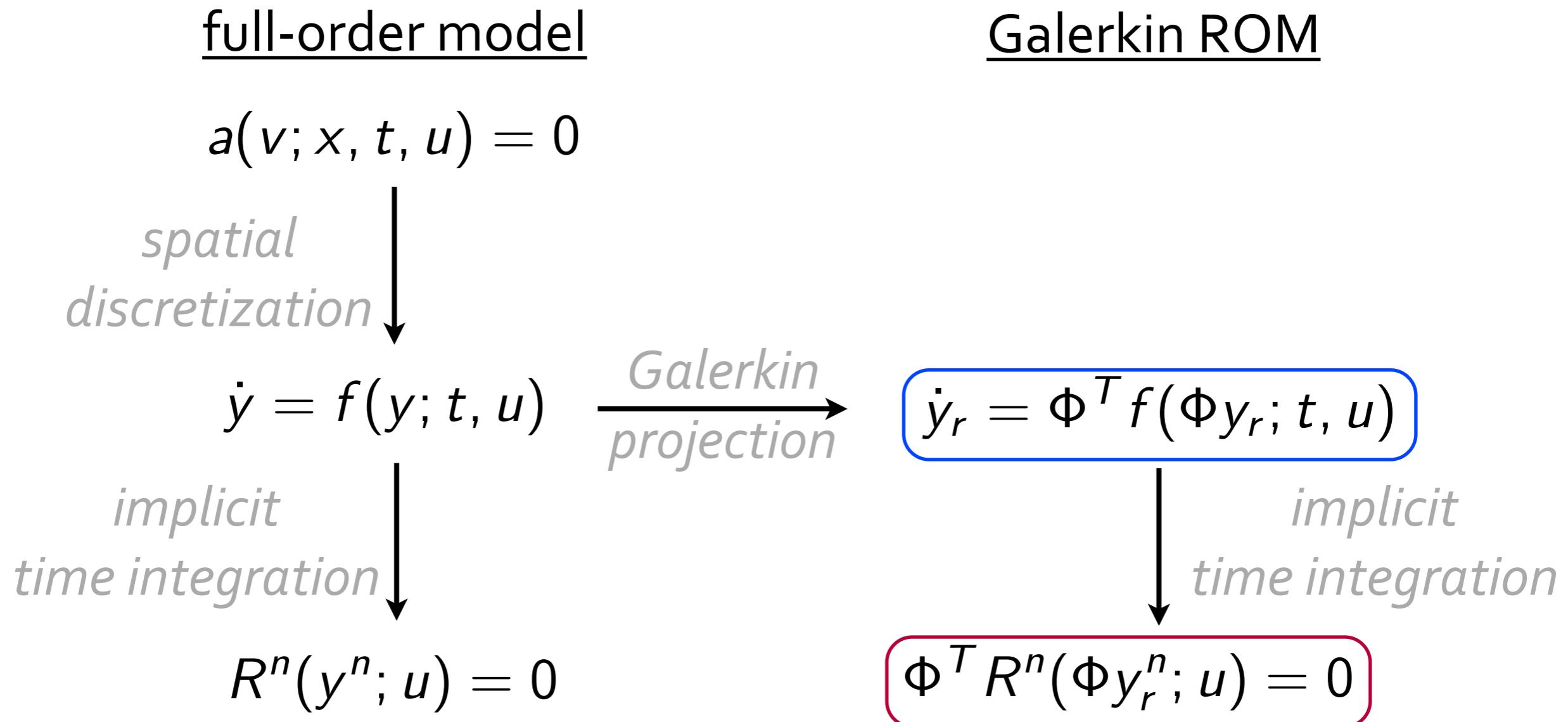
Joint work with Charbel Farhat, Julien Cortial, David Amsallem

10th World Congress on Computational Mechanics
July 12, 2012

Summary

- **Goal:** practical nonlinear model reduction
- **Problem:** POD–Galerkin often does not work
 - lacks *discrete optimality*
- **Idea:** GNAT model reduction
 - *discrete-optimal* approximations
 - effective ‘sample mesh’ implementation
 - works on large-scale problems

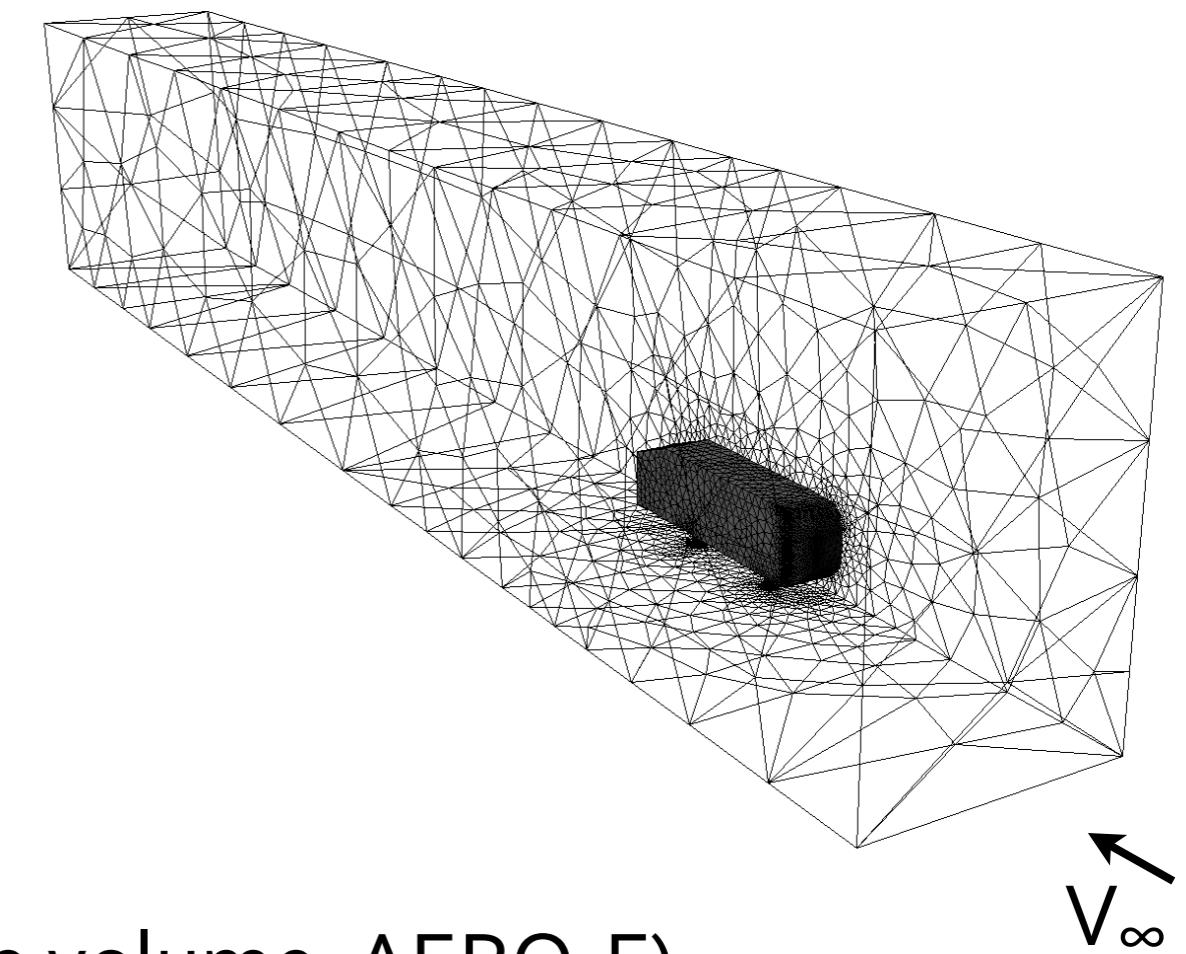
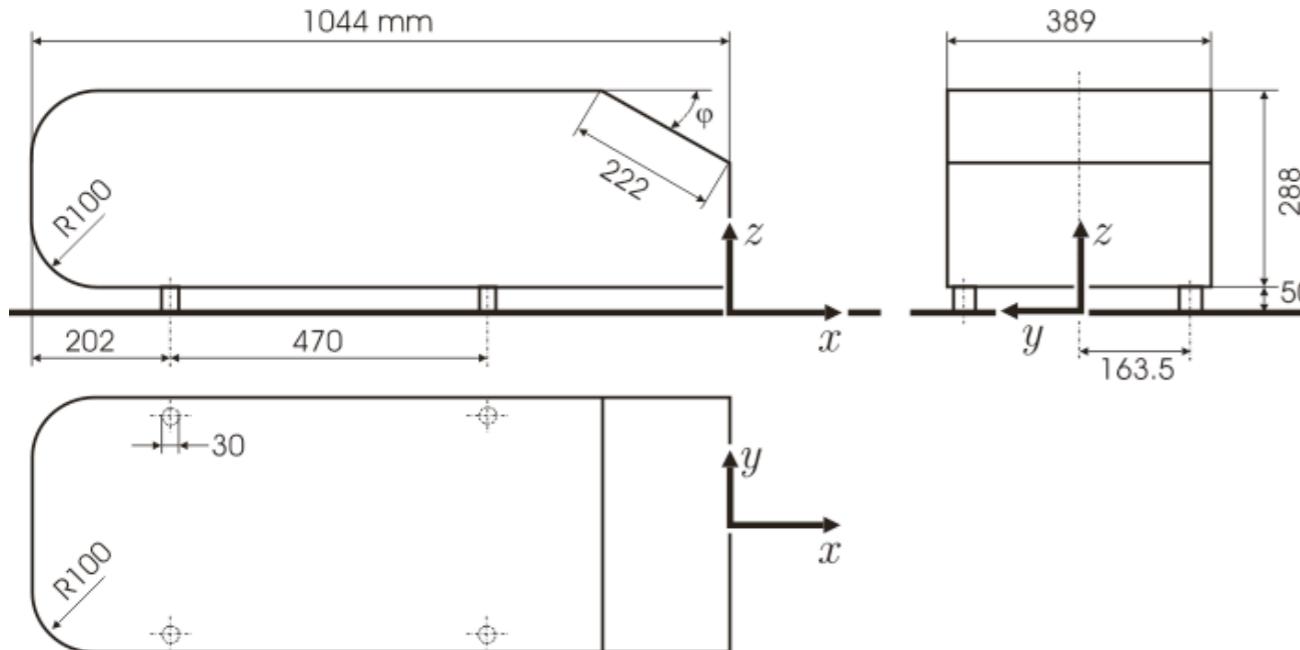
Galerkin ROM is not 'discrete optimal'



+ Semi-discrete optimal: $\Phi \dot{y}_r = \arg \min_{x \in \text{range}(\Phi)} \|\dot{y} - x\|_2$

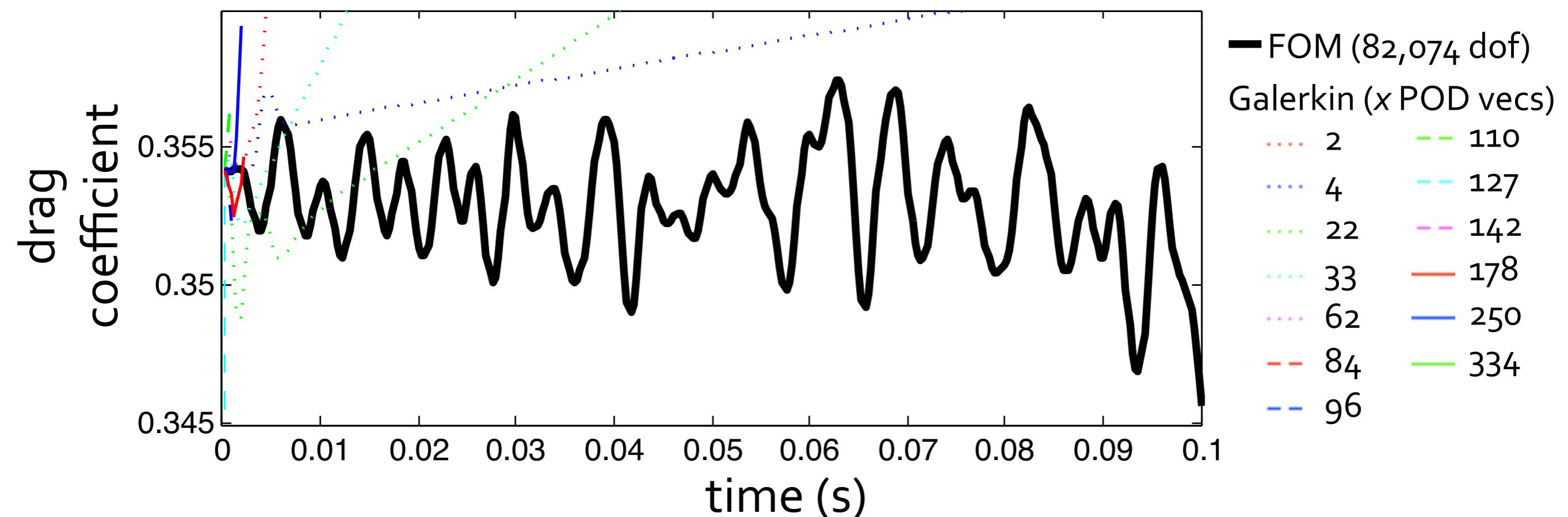
- Not discrete optimal: $\begin{cases} \Phi \dot{y}_r \neq \arg \min_{x \in \text{range}(\Phi)} \|y^n - x\| \\ \Phi y_r^n \neq \arg \min_{x \in \text{range}(\Phi)} \|R^n(x)\| \end{cases}$

Benchmark: Ahmed body



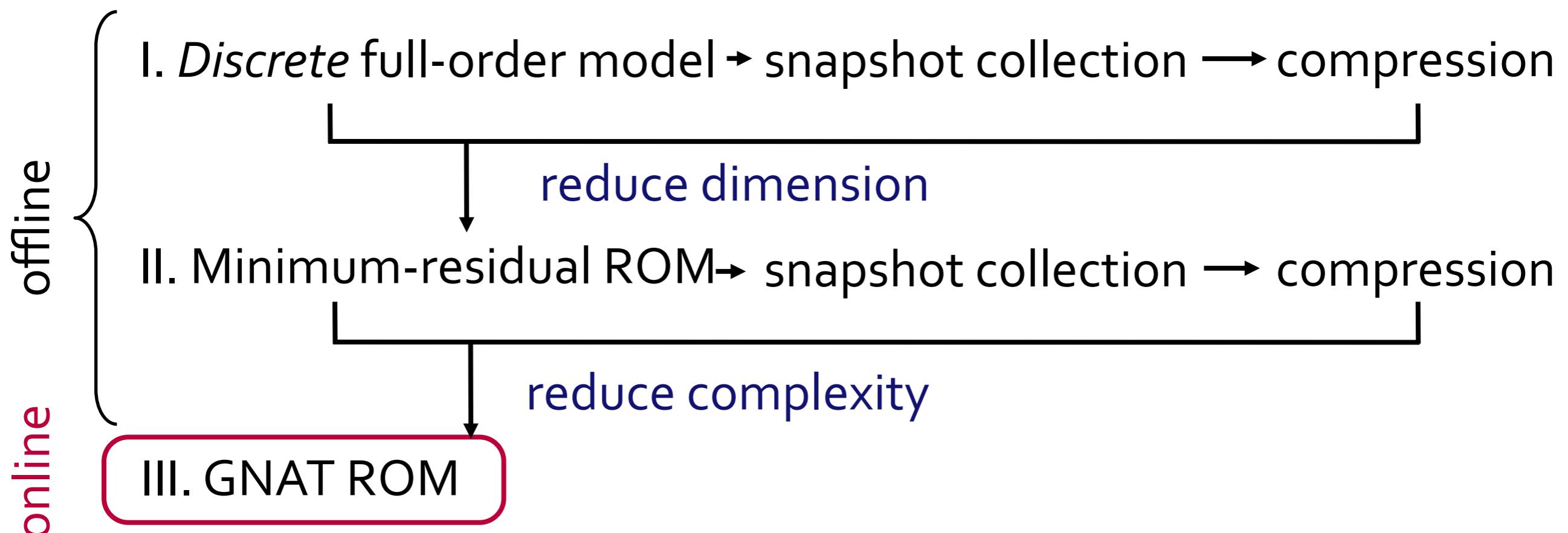
- Compressible Navier-Stokes (finite volume, AERO-F)
 - $\text{Re} = 4.48 \times 10^6$
 - $M_\infty = 0.175$ (134 mph)
 - steady-state initial condition
 - 2nd order flux reconstruction
 - DES turbulence model
 - Spalart–Allmaras RANS
 - $\Delta t = 3 \times 10^{-4}$
 - **82,074** degrees of freedom (dof)

Galerkin ROM (with POD) is unstable



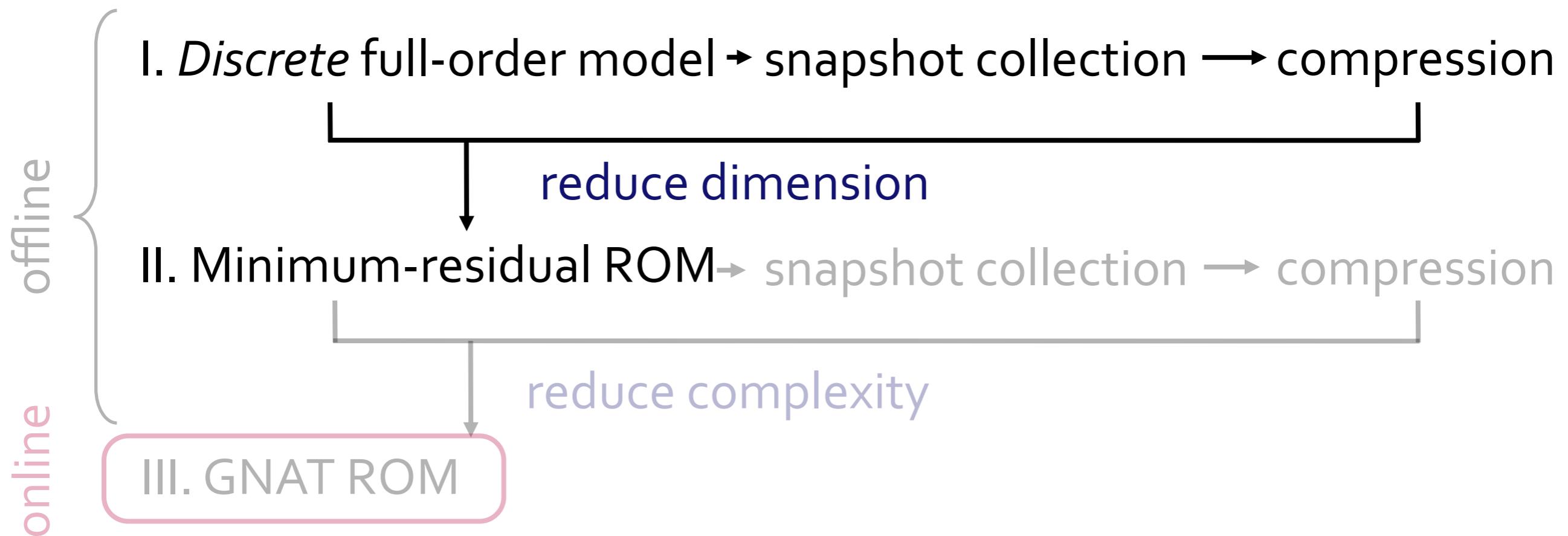
GNAT strategy

- Discrete-optimal approximations: minimize *discrete-system error* over approximation subspace



GNAT strategy

- Discrete-optimal approximations: minimize *discrete-system error* over approximation space



Minimum-residual ROM is discrete optimal

Minimum-residual ROM

not defined

full-order model

$$a(v; x, t, u) = 0$$

*spatial
discretization*



$$\dot{y} = f(y; t, u) \rightarrow$$

$$\dot{y}_r = \Phi^T f(\Phi y_r; t, u)$$

implicit

time integration



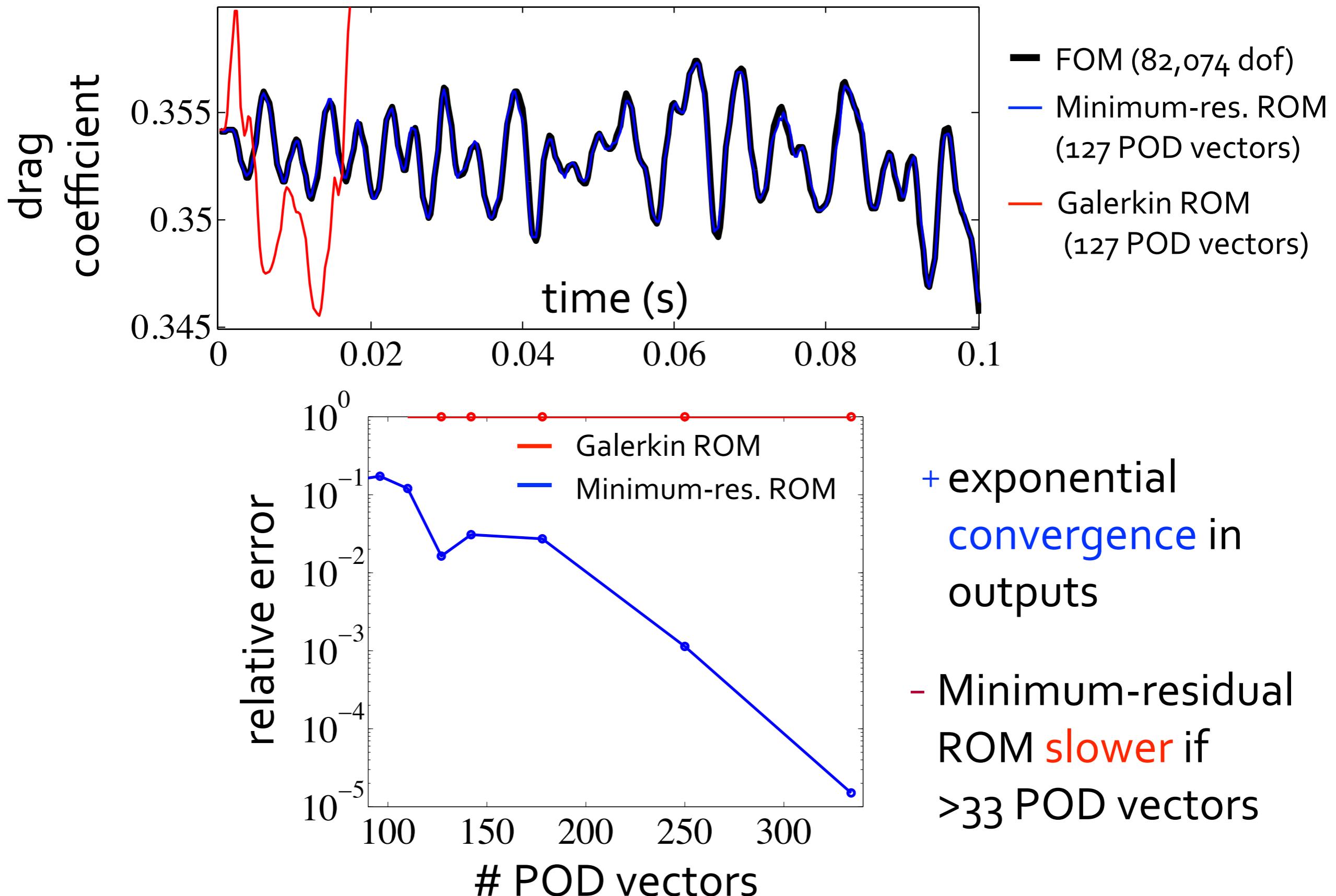
$$y_r^n = \arg \min_x \|R^n(\Phi x; u)\|_2$$

$$\leftarrow R^n(y^n; u) = 0$$

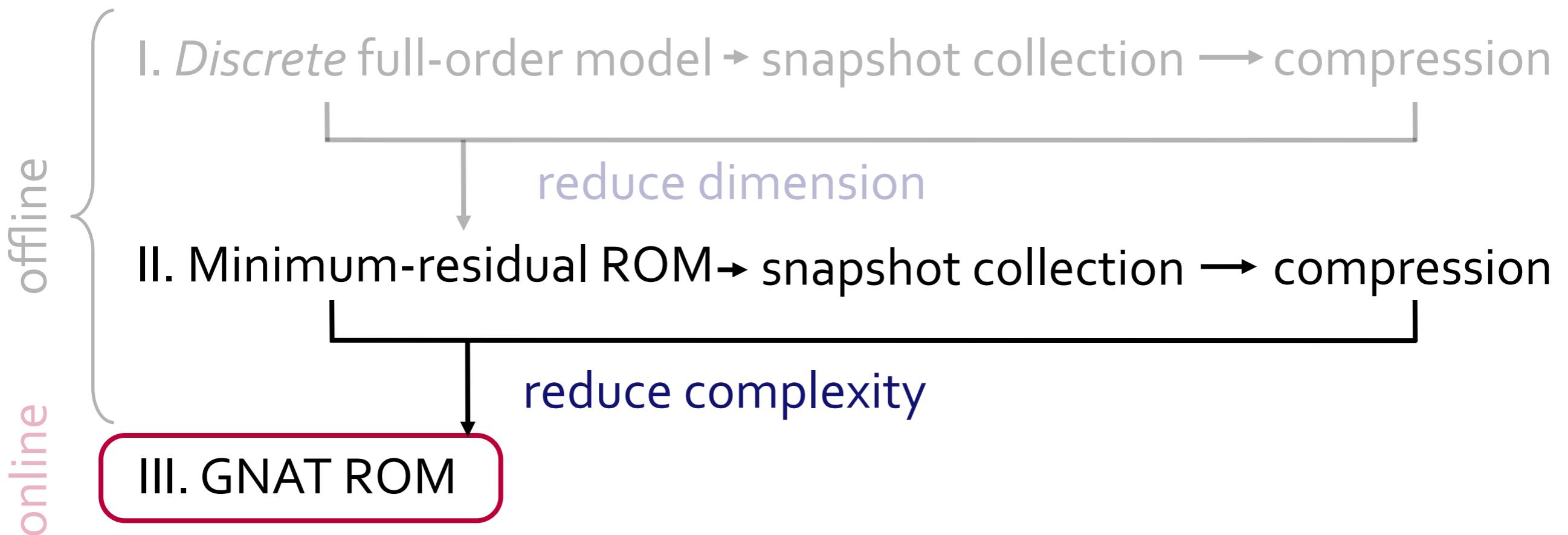
$$\Phi^T R^n(\Phi y_r^n; u) = 0$$

- + Discrete optimal
- Not defined at the semi-discrete level

Minimum-residual ROM is accurate

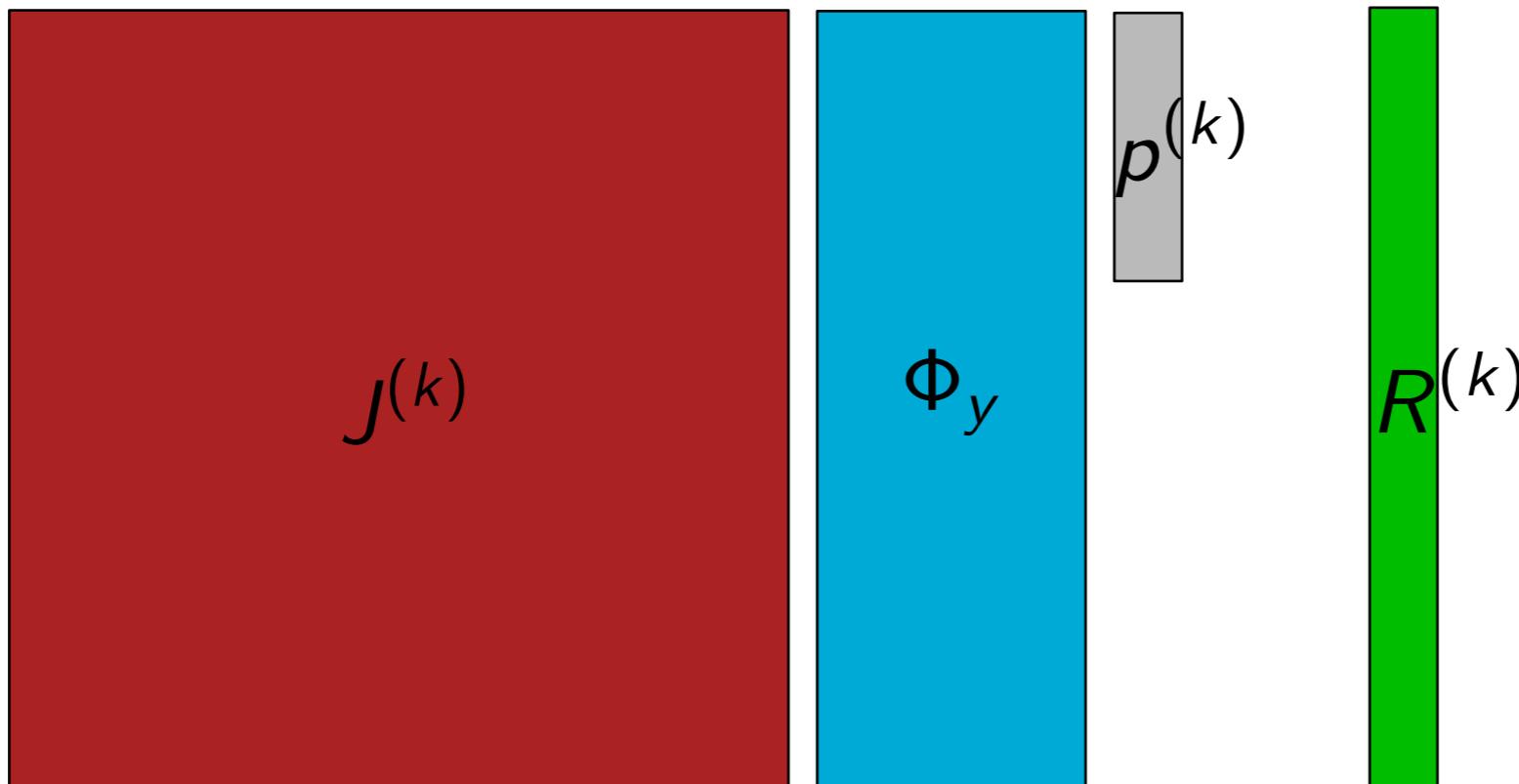


GNAT strategy



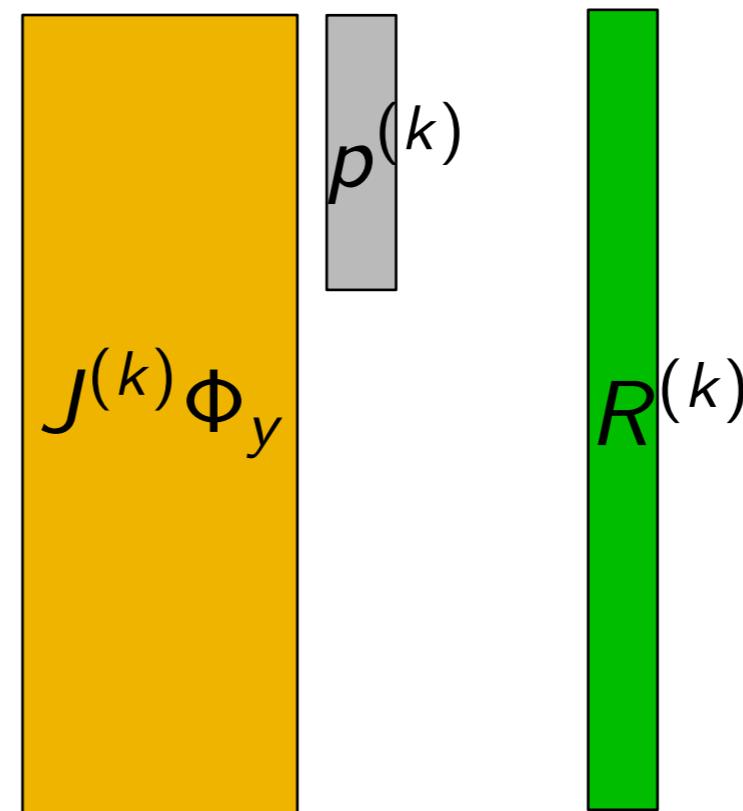
- Minimum-residual ROM iterations are

$$\underset{p^{(k)}}{\text{minimize}} \| J^{(k)} \Phi_y p^{(k)} + R^{(k)} \|_2$$

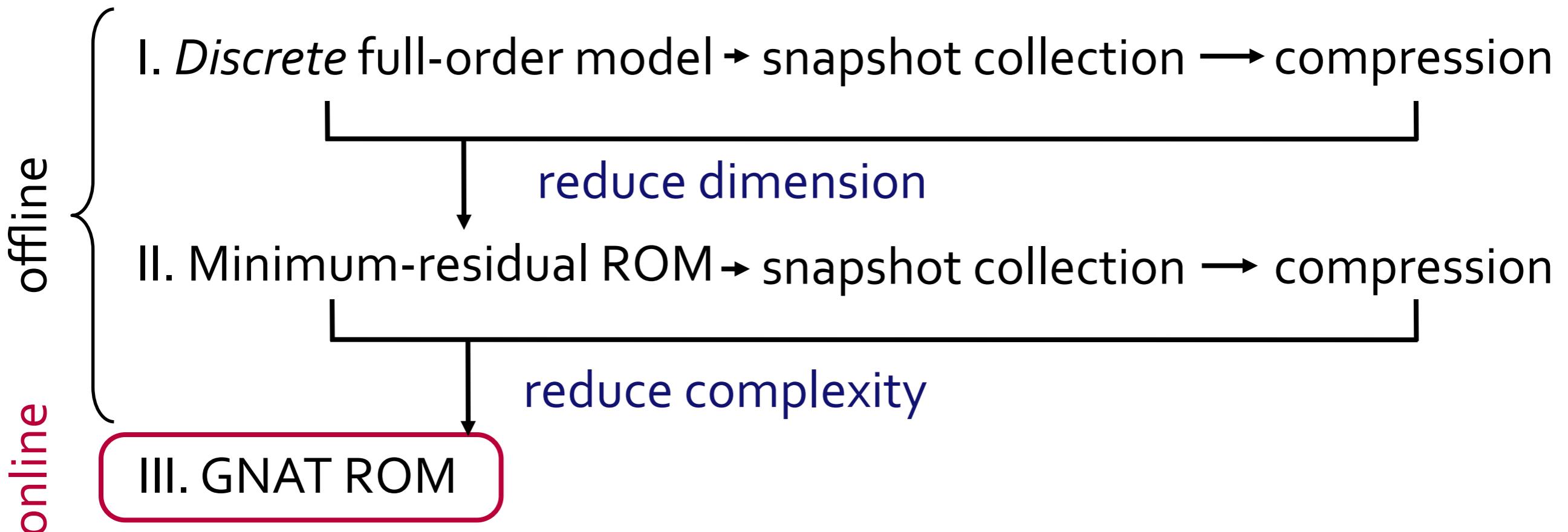


- Minimum-residual ROM iterations are

$$\underset{p^{(k)}}{\text{minimize}} \| J^{(k)} \Phi_y p^{(k)} + R^{(k)} \|_2$$



- Operation count scales with N
- Complexity reduction: sample and approximate the residual via Gappy POD [Everson & Sirovich, 1995, Carlberg et al., 2011]



GNAT = discrete-residual minimization + Gappy POD approximation

Error bound

- Assumptions:

1. backward Euler: $R^n(y^n; u) = y^n - y^{n-1} - \Delta t f(y^n; t^n, u)$
2. inverse Lipschitz continuity for $G : (y; t, u) \mapsto y - \Delta t f(y; t, u)$
3. full-order simulation convergence criterion: $\|R^n(y^n; u)\| \leq \epsilon$

Proposition [Carlberg et al., 2012]

The global error at time step n for **any sequence** $(y^0, \tilde{y}^1, \dots, \tilde{y}^{n_t})$ is

$$\|y^n - \tilde{y}^n\| \leq \sum_{k=0}^{n-1} a^k b_{n-k} \leq \sum_{k=0}^{n-1} a^k c_{n-k}$$

Proposition [Carlberg et al., 2012]

The global error at time step n for **any sequence** $(y^0, \tilde{y}^1, \dots, \tilde{y}^{n_t})$ is

$$\|y^n - \tilde{y}^n\| \leq \sum_{k=0}^{n-1} a^k b_{n-k} \leq \sum_{k=0}^{n-1} a^k c_{n-k}$$

- a : inverse Lipschitz constant ‣ \tilde{R} : residual for sequence $\{\tilde{y}^n\}$
- $b_n = \epsilon + \|\tilde{R}^n(\tilde{y}^n; u)\|$ ‣ P : Gappy POD operator
- $c_n = \epsilon + \|P\tilde{R}^n(\tilde{y}^n; u)\| + \|(I - P)\tilde{R}^n(\tilde{y}^n; u)\|$

- + Minimum-residual ROM solutions minimize b_n

Proposition [Carlberg et al., 2012]

The global error at time step n for any sequence $(y^0, \tilde{y}^1, \dots, \tilde{y}^{n_t})$ is

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 - $c_n = \epsilon + \boxed{\|P\tilde{R}^n(\tilde{y}^n; u)\|} + \|(I - P)\tilde{R}^n(\tilde{y}^n; u)\|$

+ Minimum-residual ROM solutions minimize b_n

+ GNAT ROM solutions minimize $\|P\tilde{R}^n(\tilde{y}^n; u)\|$

Proposition [Carlberg et al., 2012]

The global error at time step n for any sequence $(y^0, \tilde{y}^1, \dots, \tilde{y}^{n_t})$ is

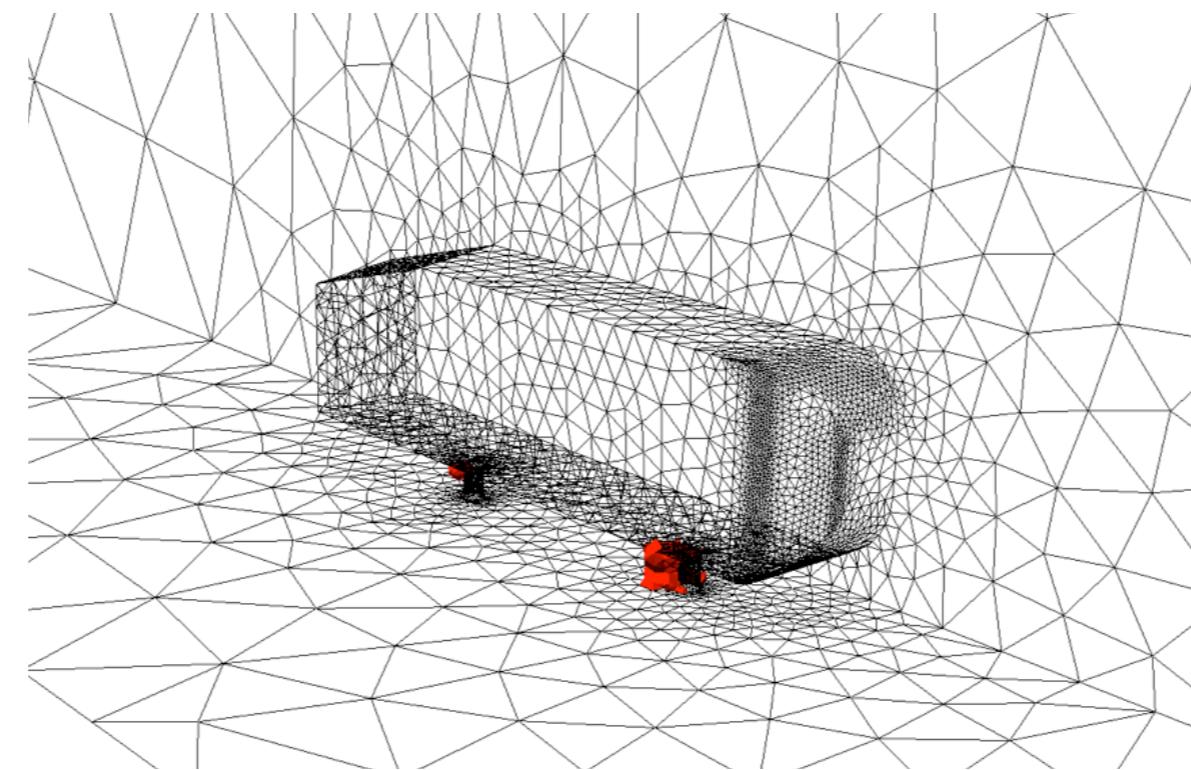
$$\|y^n - \tilde{y}^n\| \leq \sum_{k=0}^{n-1} a^k b_{n-k} \leq \sum_{k=0}^{n-1} a^k c_{n-k}$$

- a : inverse Lipschitz constant ‣ \tilde{R} : residual for sequence $\{\tilde{y}^n\}$
 - $b_n = \epsilon + \|\tilde{R}^n(\tilde{y}^n; u)\|$ ‣ P : Gappy POD operator
 - $c_n = \epsilon + \|P\tilde{R}^n(\tilde{y}^n; u)\| + \|(I - P)\tilde{R}^n(\tilde{y}^n; u)\|$

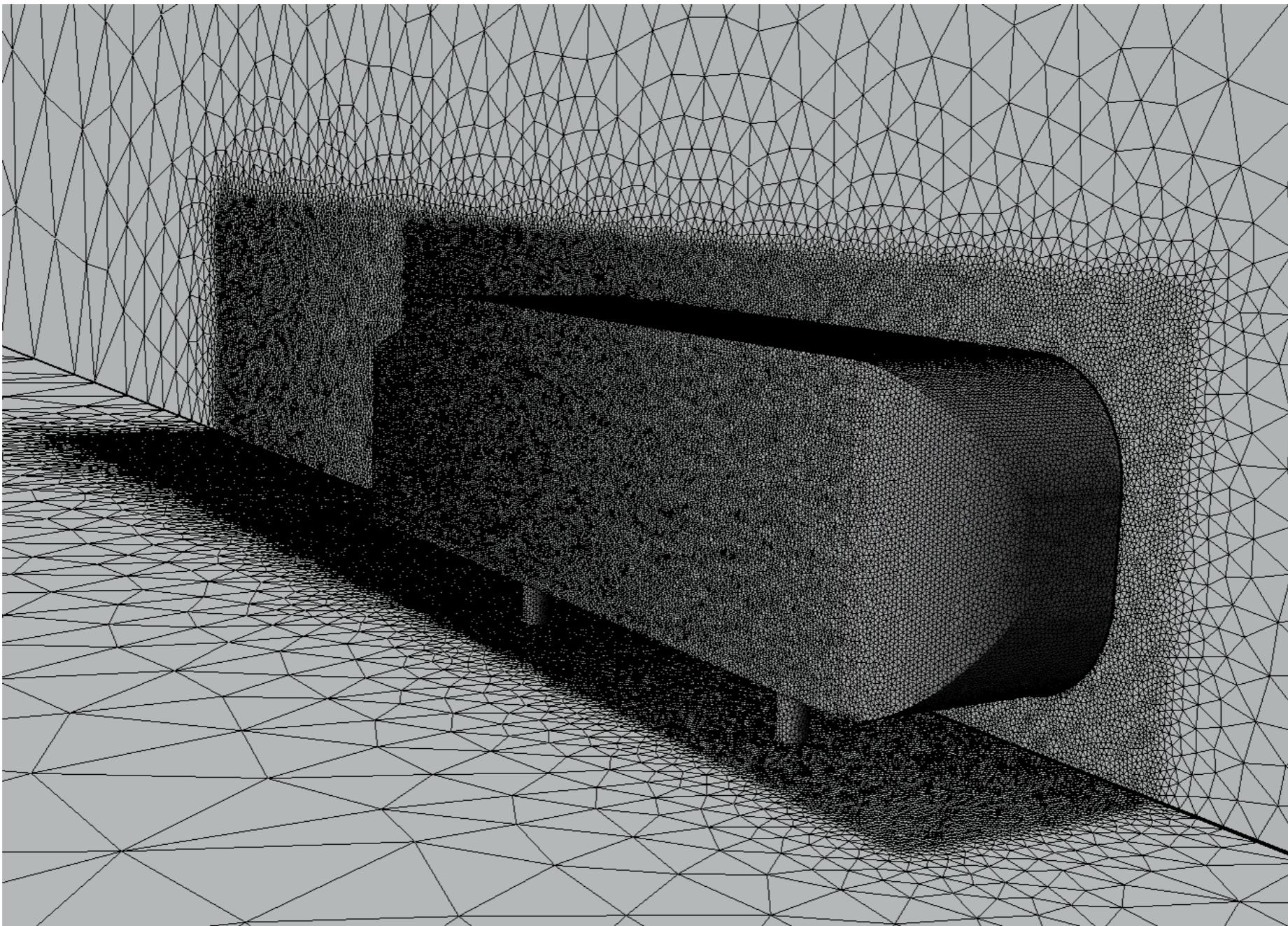
 - + Minimum-residual ROM solutions minimize b_n
 - + GNAT ROM solutions minimize $\|P\tilde{R}^n(\tilde{y}^n; u)\|$
 - + sampling algorithm: heuristic for minimizing $\|(I - P)\tilde{R}^n(\tilde{y}^n; u)\|$
 - discrete optimality enables this!

Sample mesh implementation

- *Goals:*
 - reuse existing simulation codes
 - minimize computing cores
 - scalability
- *Key:* GNAT samples only **a few** entries of the residual
- *Idea:* extract minimal subset of mesh

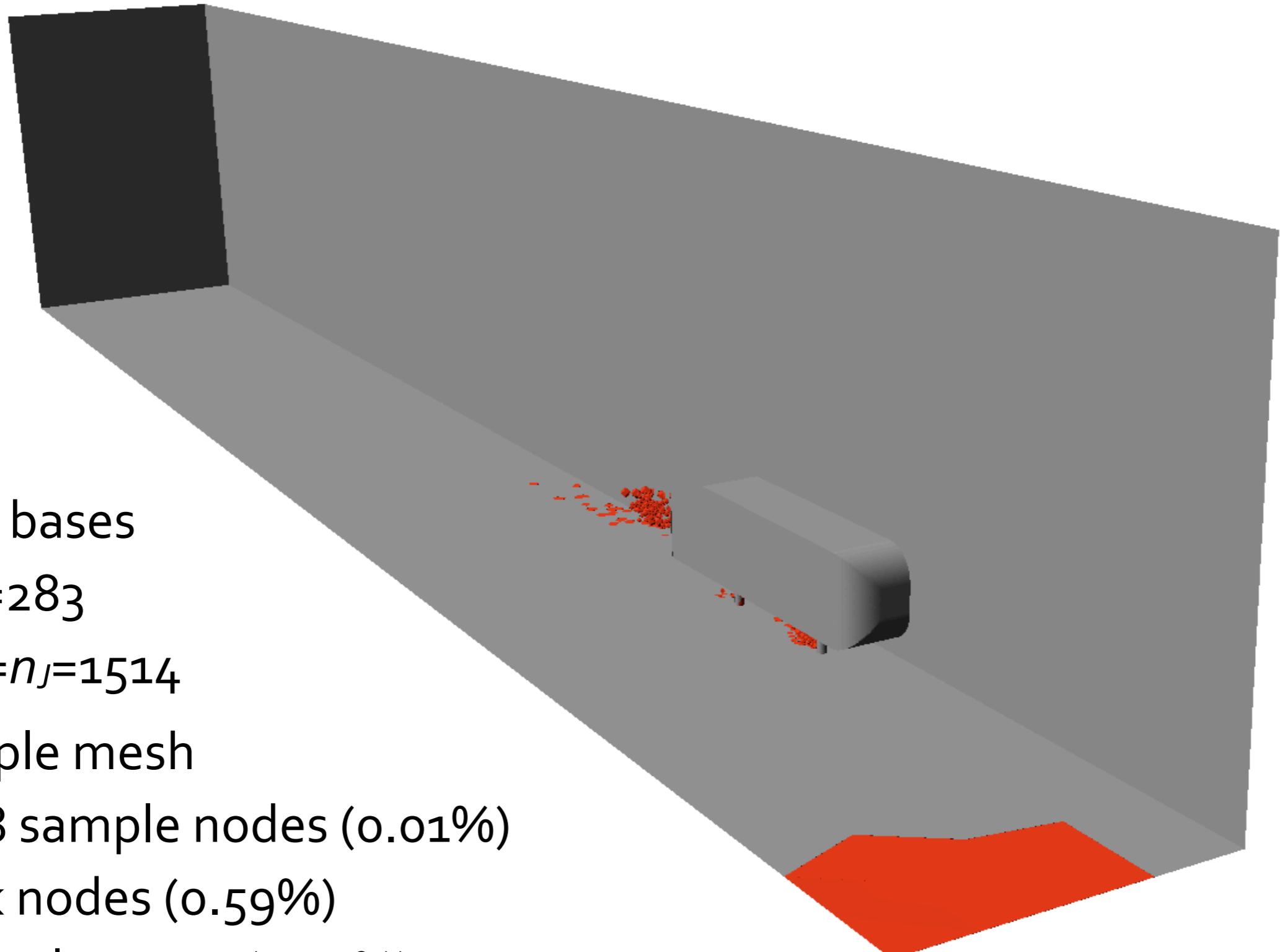


Example: Ahmed body (validated)



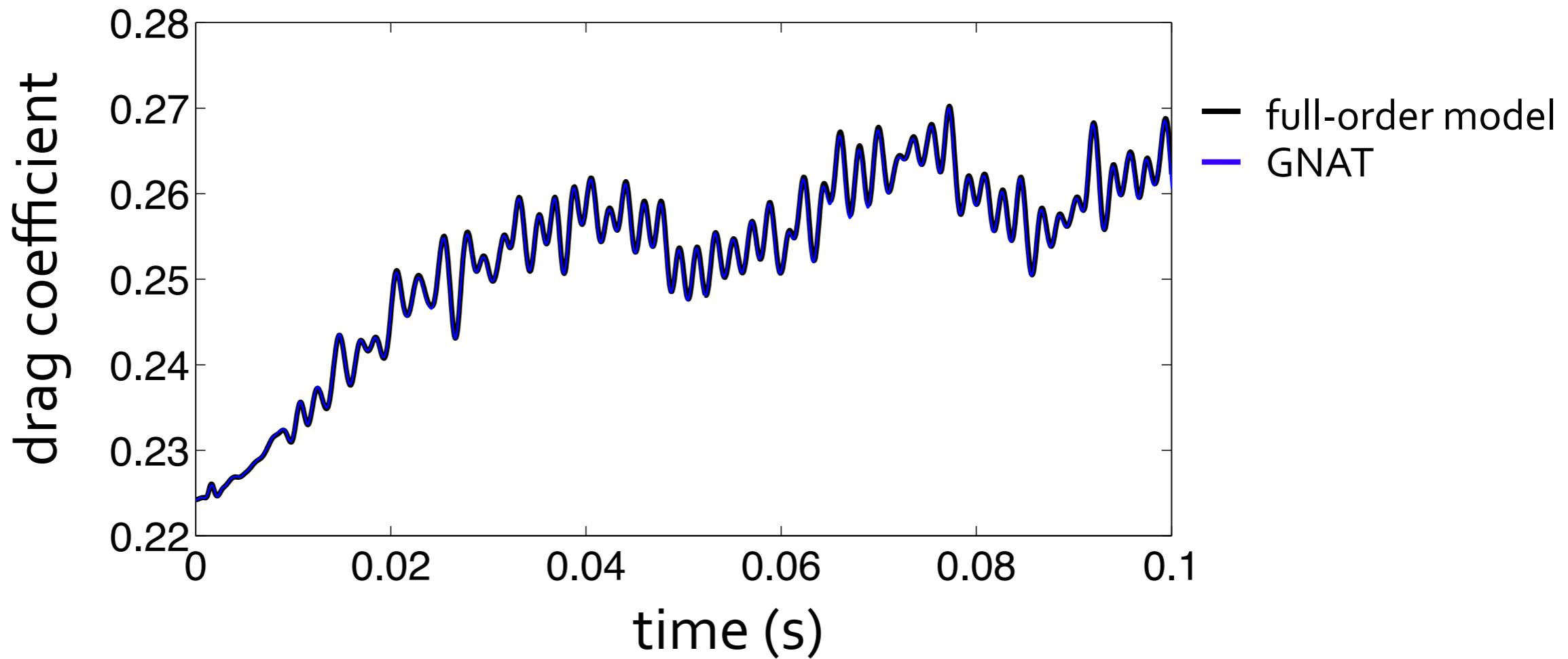
- 2.89×10^6 nodes, 1.70×10^7 tetrahedral volumes
- 1.73×10^7 degrees of freedom

GNAT model



- POD bases
 - $n_y=283$
 - $n_R=n_J=1514$
- Sample mesh
 - 378 sample nodes (0.01%)
 - 17k nodes (0.59%)
 - 56k elements (0.33%)

GNAT results: accurate and fast

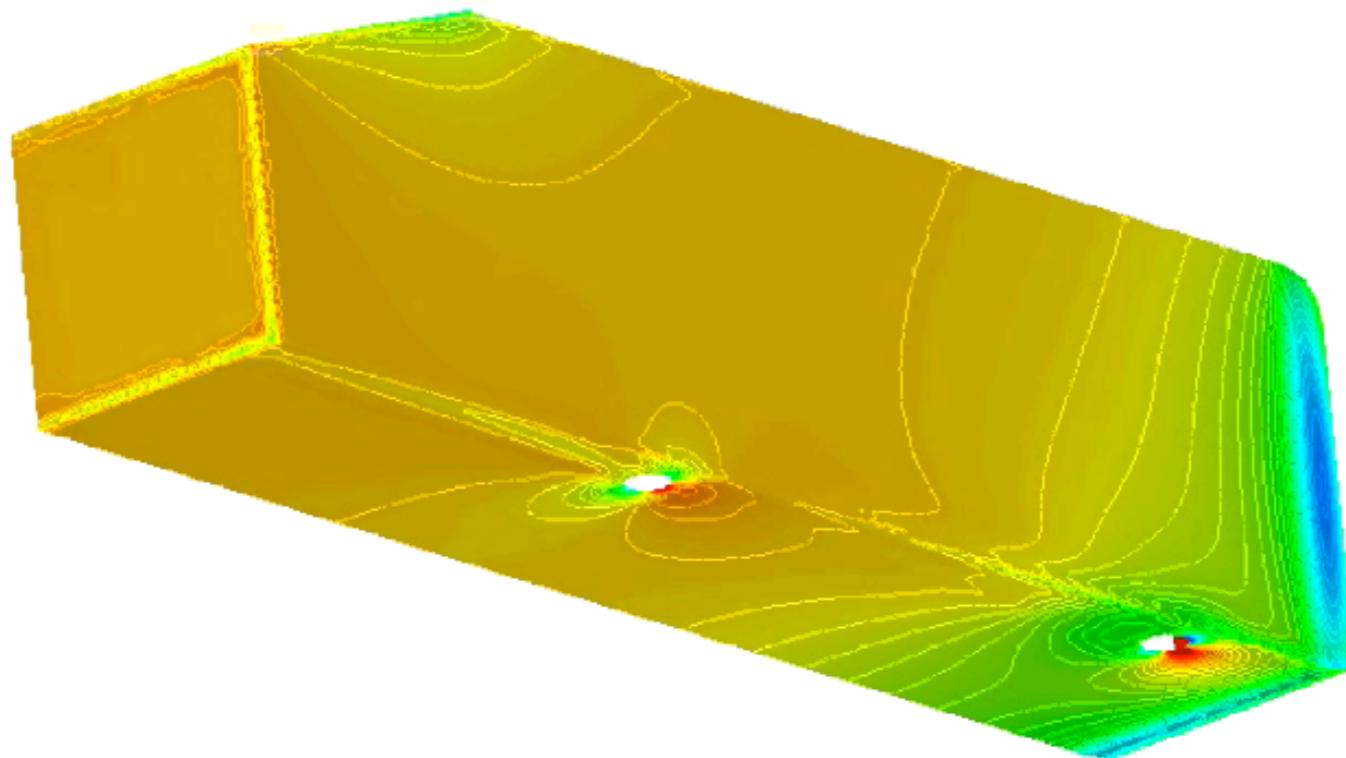


model	relative error	# cores	time, hours	speedup in cpu resources
FOM	-	512	13.3	-
GNAT	0.68%	4	3.88	438

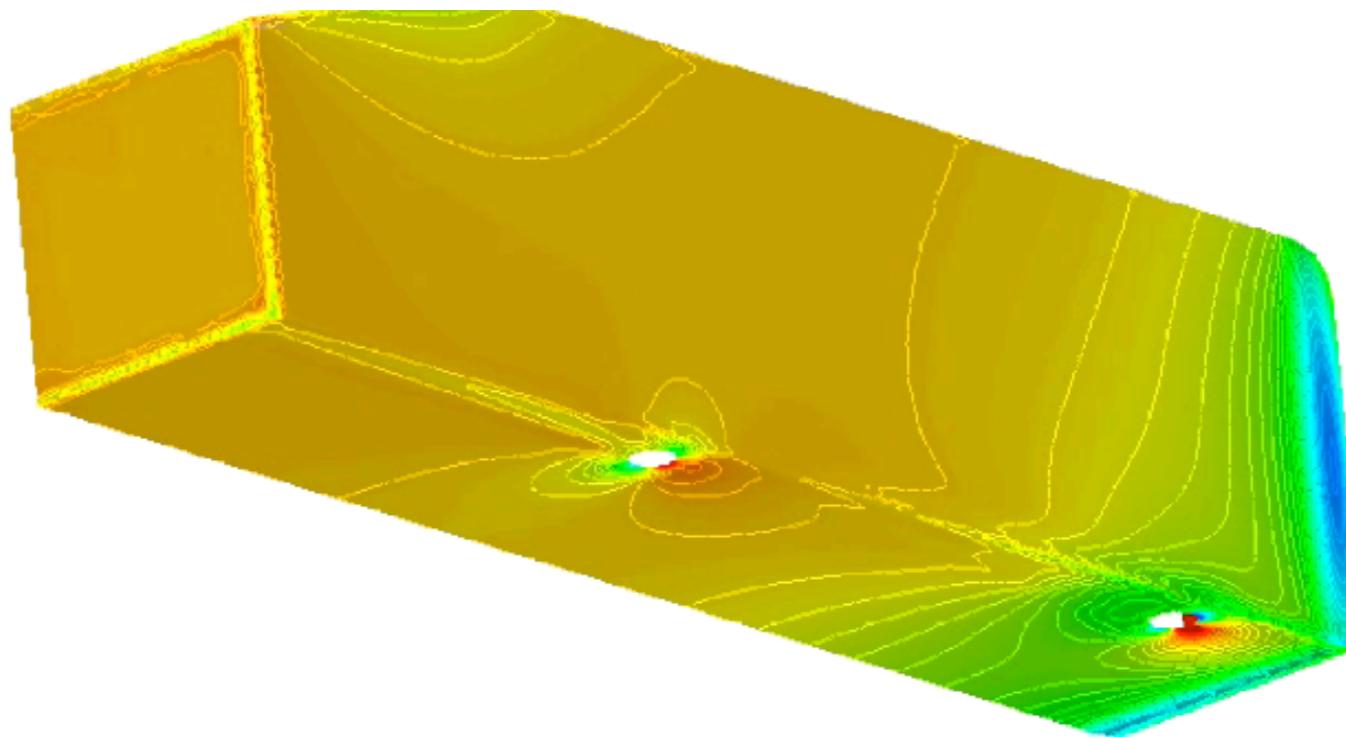
+ negligible error + wall-time decrease + supercomputer → desktop

GNAT results: accurate pressure contours

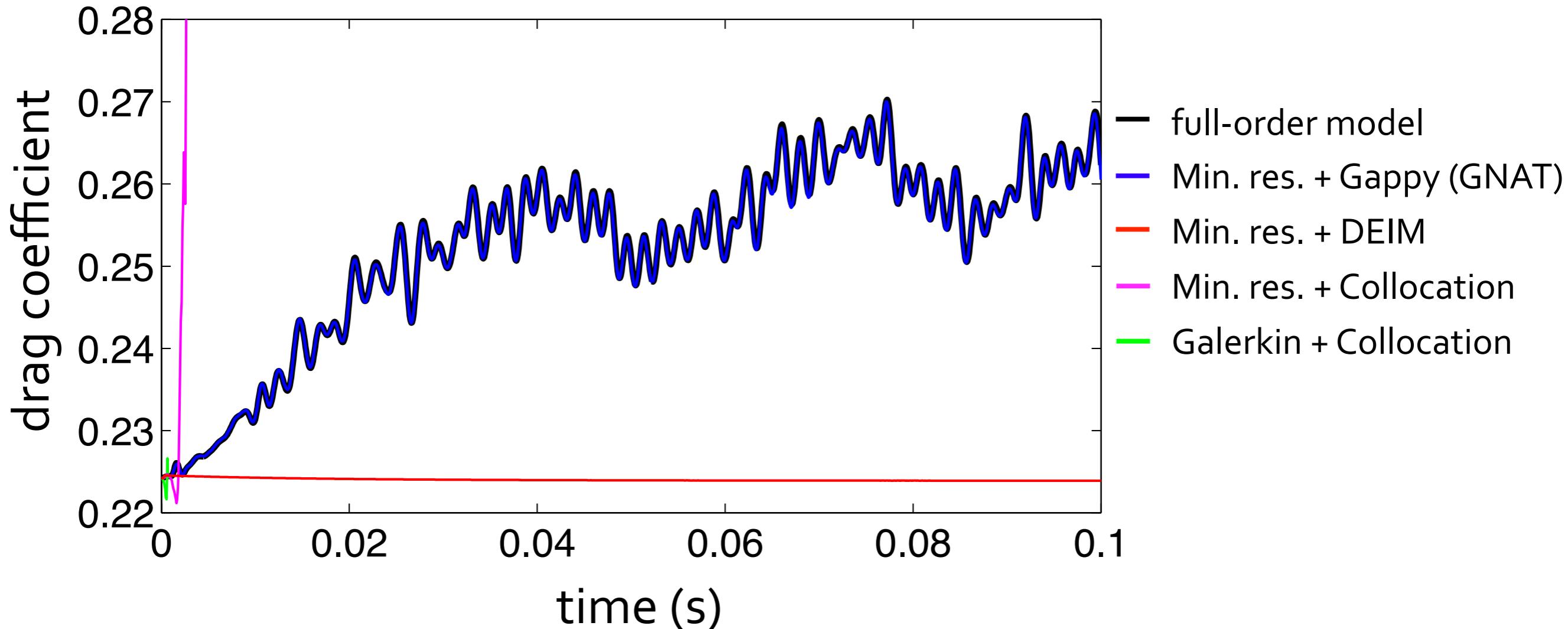
full-order
model



GNAT



- Fixed POD basis and sample mesh (378 sample nodes)



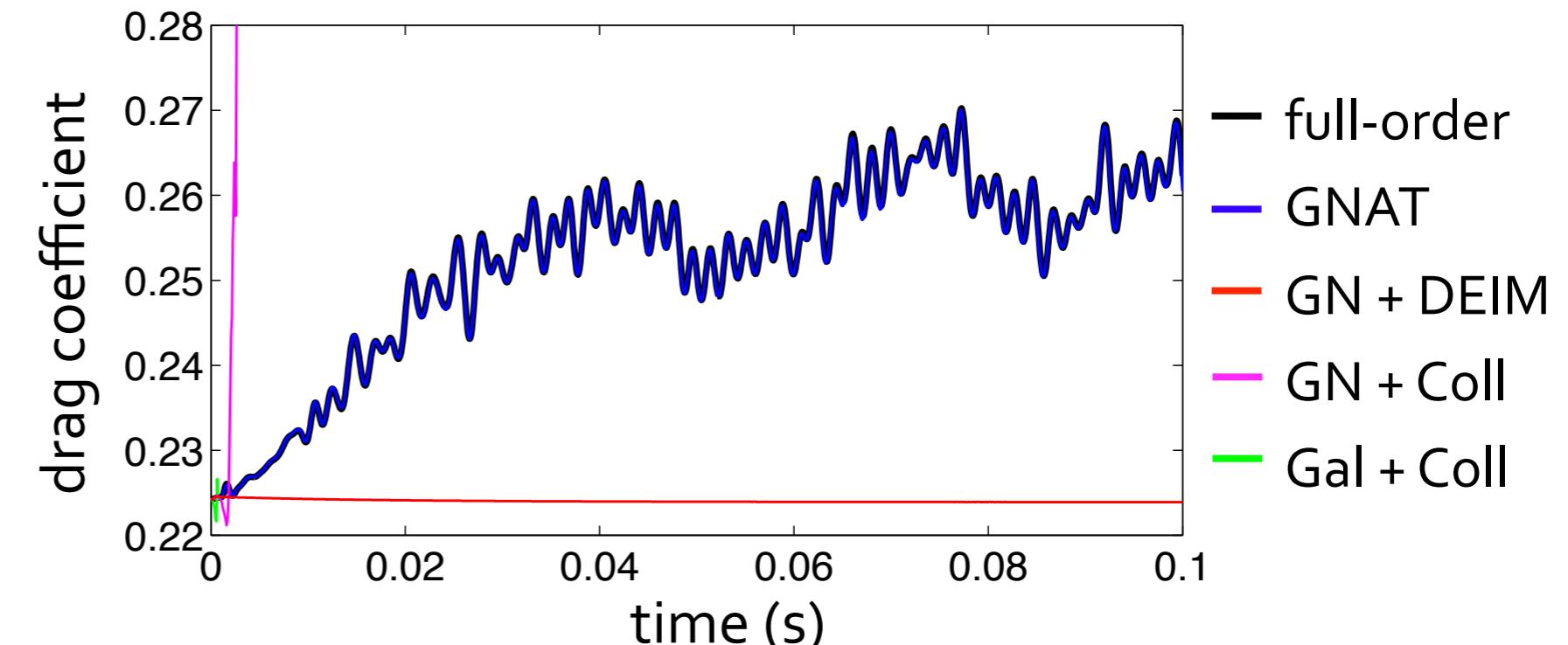
- Galerkin ROM fails
 - discrete optimality important
- Gappy POD: only complexity-reduction method that works!

Summary

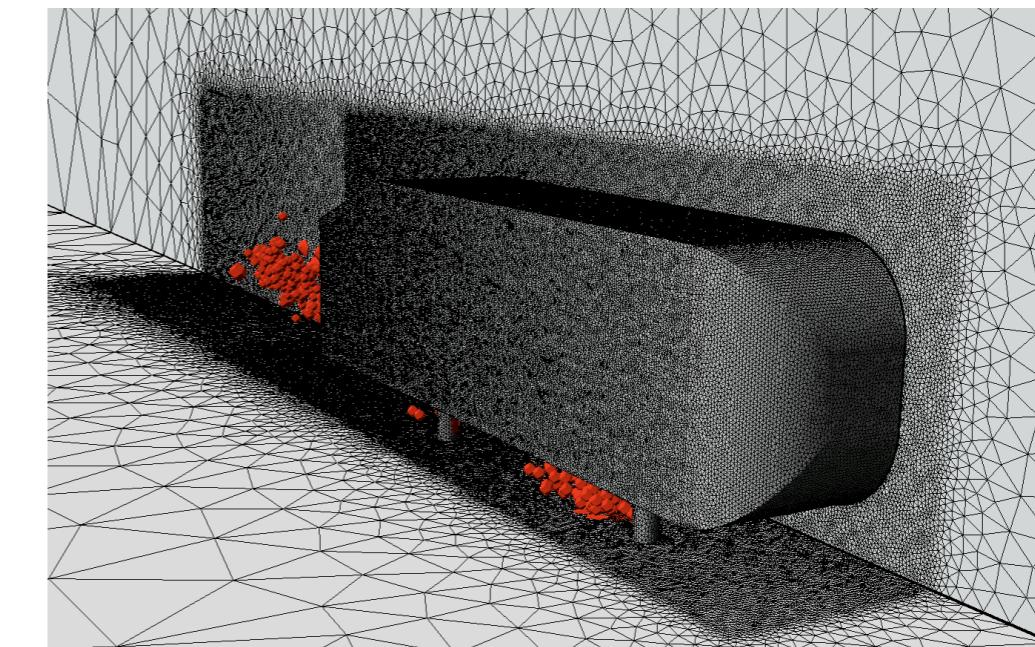
- GNAT method
 - ▶ discrete-optimal approximations
 - ▶ error bound justifies its design
- Sample mesh concept enables many fewer cores
 - ▶ supercomputer → desktop
- Ahmed body example
 - ▶ speedups over 350, error less than 1%
 - ▶ other model-reduction methods failed
- Key papers
 - ▶ K. Carlberg, C. Bou-Mosleh, and C. Farhat. "Efficient non-linear model reduction via a least-squares Petrov–Galerkin projection and compressive tensor approximations," International Journal for Numerical Methods in Engineering, Vol. 86, No. 2, p. 155–181 (2011).
 - ▶ K. Carlberg, C. Farhat, J. Cortial, and D. Amsallem, "The GNAT method for nonlinear model reduction: effective implementation and application to computational fluid dynamics and turbulent flows," *submitted*, 2012.

Questions?

- Key collaborators
 - Charbel Farhat
 - Julien Cortial
 - David Amsallem
- .



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 - NSF Graduate Fellowship
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 - Army Research Laboratory
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